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Nodes in the Relativistic Quantum Trajectories and Photon's Trajectories

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Abstract

Through the constant potential and the linear potential, we establish the existence of nodes for the relativistic quantum trajectories as the same way as for the quantum trajectories. We establish the purely relativistic limit ($\hbar \rightarrow 0$) for these trajectories, and link the nodes to de Broglie's wavelength.

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1. Introduction

Recently, we have presented with Bouda a new formulation of quantum mechanics based on both new quantum Lagrangian and the solution of the quantum stationary Hamilton-Jacobi equation (QSHJE) [1]. The Bouda and Djama Lagrangian lead to the quantum Newton's law, from which they derived and plotted the quantum trajectories (QT) for several potentials [2]. In a previous paper [3], we have generalized the formulation presented in Ref. [1, 2] into the relativistic quantum systems. We started from the relativistic quantum stationary Hamilton-Jacobi equation (RQSHJE) presented by Faraggi and Matone in Refs. [4, 5, 6] as

$$\frac{1}{2m_0} \left(\frac{\partial S_0}{\partial x} \right)^2 - \frac{\hbar^2}{4m_0} \left[\frac{3}{2} \left(\frac{\partial S_0}{\partial x} \right)^{-2} \left(\frac{\partial^2 S_0}{\partial x^2} \right)^2 - \left(\frac{\partial S_0}{\partial x} \right)^{-1} \left(\frac{\partial^3 S_0}{\partial x^3} \right) \right] + \frac{1}{2m_0 c^2} [m_0^2 c^4 - (E - V)^2] = 0, \quad (1)$$

where S_0 represent the relativistic quantum reduced action, E the total energy including the energy at rest, V the potential and m_0 the mass at rest of the particle. c is the light velocity in vacuum. The relativistic quantum reduced action is given by [1, 4, 5, 6]

$$S_0 = \hbar \arctan \left[a \frac{\phi_1}{\phi_2} + b \right]. \quad (2)$$

a and b are real integration constants. ϕ_1 and ϕ_2 are two real independents solutions of the Klein-Gordon equation

$$-c^2 \hbar^2 \frac{\partial^2 \phi}{\partial x^2} + [m_0^2 c^4 - (E - V)^2] \phi(x) = 0. \quad (3)$$

The conjugate momentum can be written following Eq. (2) as

$$\frac{\partial S_0}{\partial x} = \pm \frac{\hbar a W}{\phi_2^2 + (a\phi_1 + b\phi_2)^2}. \quad (4)$$

W represent the wronskian of ϕ_1 and ϕ_2 with respect to x . The sign \pm in Eq. (4) represent the two possibilities of motion: x positive or x negative directions.

Taking advantage on this results, we introduced in Ref. [3] a relativistic quantum Lagrangian

$$L = -m_0 c^2 \sqrt{1 - \frac{\dot{x}^2}{c^2}} f(x, \mu, \nu, E) - V(x), \quad (5)$$

from which we derived the first integral of the relativistic quantum Newton's Law (FIRQNL).

$$\begin{aligned} & [(E - V)^2 - m_0^2 c^4]^2 + \frac{\dot{x}^2}{c^2} (E - V)^2 [(E - V)^2 - m_0^2 c^4] + \frac{\hbar^2}{2} \left[\frac{3}{2} \left(\frac{\ddot{x}}{\dot{x}} \right)^2 - \frac{\ddot{x}}{\dot{x}} \right] \cdot \\ & (E - V)^2 - \frac{\hbar^2}{2} \left(\ddot{x} \frac{dV}{dx} + \dot{x}^2 \frac{d^2 V}{dx^2} \right) \left[\frac{(E - V)^2 + m_0^2 c^4}{(E - V)^2 - m_0^2 c^4} \right] (E - V)^2 - \frac{3\hbar^2}{4} \cdot \\ & \left(\dot{x} \frac{dV}{dx} \right)^2 \left[\frac{(E - V)^2 + m_0^2 c^4}{(E - V)^2 - m_0^2 c^4} \right]^2 - \hbar^2 \left(\dot{x} \frac{dV}{dx} \right)^2 \frac{m_0^2 c^4}{(E - V)^2 - m_0^2 c^4} = 0. \end{aligned} \quad (6)$$

We also established that the conjugate momentum of relativistic quantum systems is connected with the particle velocity as follows

$$\frac{\partial S_0}{\partial x} = \frac{E - V(x)}{\dot{x}} - \frac{m_0^2 c^4}{(E - V)\dot{x}}, \quad (7)$$

From Eqs. (4) and (7), one can write the following expressions

$$\frac{dx}{dt} = pm \frac{1}{\hbar a W} \left[E - V(x) - \frac{m_0^2 c^4}{E - V(x)} \right] [\phi_2^2 + (a\phi_1 + b\phi_2)^2]. \quad (8)$$

We choose the sign of aW in such a way that in classically allowed region ($E - V > m_0 c^2$), in Eq. (8) dx/dt takes the positive sign for x positive directions and the negative sign for the x negative directions. Furthermore, at the turning point dx/dt changes the sign either the particle goes in classically allowed regions or in the classically forbidden regions.

2. Motion of electron under the constant potential

In the case of an electron moving under a constant potential $V = U_0$, the FIRQNL (Eq. (6)) and the conjugate momentum (Eq. (7)) reduce to the following equations

$$[(E - U_0)^2 - m_0^2 c^4]^2 - \frac{\dot{x}^2}{c^2} (E - U_0)^2 [(E - U_0)^2 - m_0^2 c^4] + \frac{\hbar^2}{2} \left[\frac{3}{2} \left(\frac{\ddot{x}}{\dot{x}} \right)^2 - \frac{\dot{\ddot{x}}}{\dot{x}} \right] (E - U_0)^2 = 0. \quad (9)$$

and

$$\dot{x} \frac{\partial S_0}{\partial x} = E - U_0 - \frac{m_0^2 c^4}{E - U_0}. \quad (10)$$

The solution of these equation is

$$x(t) = \frac{\hbar c}{\sqrt{E^2 - m_0^2 c^4}} \arctan \left[a \tan \left(\frac{E^2 - m_0^2 c^4}{\hbar E} t \right) + b \right] + \frac{\pi \hbar c}{\sqrt{E^2 - m_0^2 c^4}} n + x_0. \quad (11)$$

with

$$t \in \left[\frac{\pi \hbar E}{E^2 - m_0^2 c^4} \left(n - \frac{1}{2} \right); \frac{\pi \hbar E}{E^2 - m_0^2 c^4} \left(n + \frac{1}{2} \right) \right]$$

for every integer number. In Eq. (11) we added the term multiple of n to assure that the particle will not contained between

$$-\frac{\hbar c}{\sqrt{(E - U_0)^2 - m_0^2 c^4}} \frac{\pi}{2} + x_0$$

and

$$\frac{\hbar c}{\sqrt{(E - U_0)^2 - m_0^2 c^4}} \frac{\pi}{2} + x_0.$$

Note that for $a = 1$ and $b = 0$ the relation (11) reduces to the relativistic relation

$$x(t) = \frac{c}{E - U_0} \sqrt{(E - U_0)^2 - m_0^2 c^4} t + x_0. \quad (12)$$

In Fig. 1, we have plotted in (t, x) plane for an electron of energy $2MeV + U_0$ some trajectories for different values of a and b . All these trajectories pass

$x (\times 10^{-12} \text{ m})$

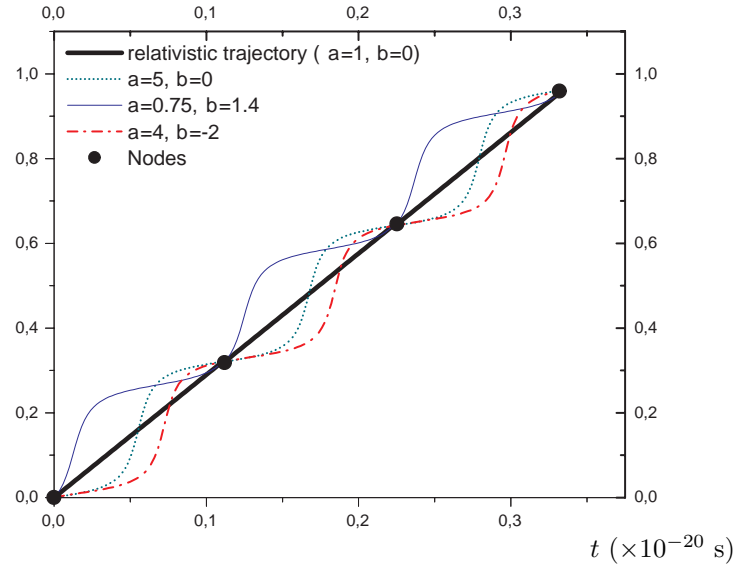


Fig. 1: Relativistic quantum trajectories for a free electron of energy $E = 2 \text{ MeV}$. For all the curves, we have chosen $x(t=0) = 0$.

through nodes exactly as we have seen for quantum trajectories of an electron moving under a constant potential [3]. These nodes correspond to the times

$$t_n = \frac{\pi \hbar E - U_0}{(E - U_0)^2 - m_0^2 c^4} \left(n + \frac{1}{2} \right). \quad (13)$$

The distance between two adjacent nodes is on time axis

$$\Delta t_n = t_{n+1} - t_n = \frac{\pi \hbar (E - U_0)}{(E - U_0)^2 - m_0^2 c^4}. \quad (14)$$

and space axis

$$\Delta x_n = x_{n+1} - x_n = \frac{\pi \hbar c}{\sqrt{(E - U_0)^2 - m_0^2 c^4}}. \quad (15)$$

These distances are both proportional to \hbar meaning that at the classical limit $\hbar \rightarrow 0$ the nodes become infinitely near, and then, all possible relativistic quantum trajectories tend to the purely relativistic one [1, 2]. Indeed, let us consider an arbitrary point $P(x, t)$ from any relativistic trajectory. Obviously, this point is situated between two nodes. Now, if we take the orthogonal projection P_0 of P on the relativistic trajectory ($a = 1$, $b = 0$) and compute the distance PP_0 we get

$$PP_0 = c \sqrt{2 - \frac{(E - U_0)^2 - m_0^2 c^4}{(E - U_0)^2}} \|t_P - t_{P_0}\|. \quad (16)$$

Since Eq. (8) indicates that \dot{x} is a monotonous function, $\|t_P - t_{P_0}\| < t_{n+1} - t_n$ for every relativistic quantum trajectory [2]. Then at the classical limit ($\hbar \rightarrow 0$), $\|t_P - t_{P_0}\| \rightarrow 0$ and $PP_0 \rightarrow 0$. Thus, all quantum trajectories goes when ($\hbar \rightarrow 0$) to the purely relativistic one. This result is with accord with the fact that our dynamical equations (Eqs (1), (6) and (7)) tend to the relativistic equation when ($\hbar \rightarrow 0$).

Now let us consider the problem when $E - U_0 < 0$ when the electron is in the classically forbidden regions. For this case the solution of Eqs. (9) and (10) is

$$x(t) = \frac{\hbar c}{2\sqrt{m_0^2 c^4 - (E - U_0)^2}} \ln \left| a \tan \left(\frac{m_0^2 c^4 - (E - U_0)^2}{\hbar E - U_0} t \right) + b \right| + x_0. \quad (17)$$

where a , b and c are real integration constant with $a \neq 0$. We see clearly from Eq. (17) that at the finite time $-(2n+1)\pi\hbar E/4((E-U_0)^2 - m_0^2 c^4)$ the electron cross an infinite distance and reach an infinite speed. This is in accordance with standard quantum tunneling theories which predict infinite velocities and finite reflexion times for tunneling phenomena (Fletcher [7], Hartman [8]).

In Fig. (2), we plotted for an electron of energy $2\text{Mev} - U_0$ a relativistic quantum trajectory in the classically forbidden regions with $a = 4$ and $b = 2$. This figure shows clearly how the particle reach an infinite position at a finite time. In particular, as it is shown by Eq. (17), classically forbidden regions there is no nodes for the RQT.

3. Photon trajectories

We propose, now, to study the motion of a photon in a presence of a constant potential U_0 . For this aim, we must take into consideration that the photon's

$x (\times 10^{-12} \text{ m})$

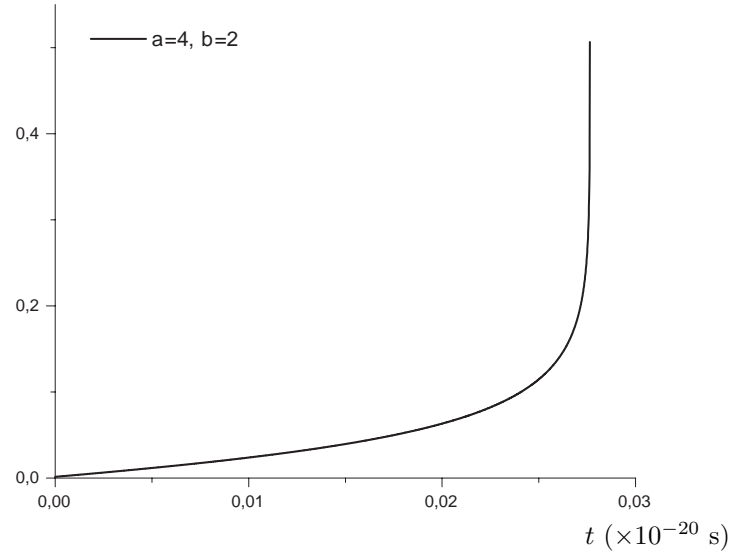


Fig. 2: Relativistic quantum trajectories for an electron of energy $E = 2 - U_0$ Mev moving in the classically forbidden regions. We have chosen $x(t=0) = 0$.

mass at rest is null, otherwise its relativistic mass were be infinite. Then, Eqs. (6) and (7) can be written as

$$(E - U_0)^2 - \dot{x}^2 \frac{(E - U_0)^2}{c^2} + \frac{\hbar^2}{2} \left[\frac{3}{2} \left(\frac{\ddot{x}}{\dot{x}} \right)^2 - \frac{\dot{\ddot{x}}}{\dot{x}} \right] = 0 . \quad (18)$$

and

$$\dot{x} \frac{\partial S_0}{\partial x} = E - U_0 . \quad (19)$$

Before introducing the solution of Eq. (18), note that Eq. (19) indicates that \dot{x} and $\partial S_0 / \partial x$ have the same sign in classically permitted regions, and are opposite in classically forbidden regions. Then, Eq. (19) reduces, after using the solutions $\cos \left(\frac{\|E - U_0\|}{\hbar c} x \right)$ and $\sin \left(\frac{\|E - U_0\|}{\hbar c} x \right)$ of Eq. (3), to

$$\frac{dx}{dt} = \pm \frac{c}{a} \left[\cos^2 \left(\frac{\|E - U_0\|}{\hbar c} x \right) + \left[a \sin \left(\frac{\|E - U_0\|}{\hbar c} x \right) + b \cos \left(\frac{\|E - U_0\|}{\hbar c} x \right) \right]^2 \right] , \quad (20)$$

where the sign $+$ is used for the classically permitted regions and the sign $-$ is used for the forbidden.

As we can deduce from Eq. (20) the motion of a photon under a constant potential is identical in the two quantum possible situations, the classically forbidden regions and permitted regions. It is essentially due to the symmetry that the Klein-Gordon equation shows for the photon since its solutions are trigonometric equation either for $E - U_0 > 0$ or $E - U_0 < 0$. The resolution of Eq. (20) gives [3]

$$x(t) = \frac{\hbar c}{\|E - U_0\|} \arctan \left[a \tan \left(\frac{\|E - U_0\|}{\hbar} t \right) + b \right] + \frac{\pi \hbar c}{\|E - U_0\|} n + x_0 , \quad (21)$$

with

$$t \in \left[\frac{\pi \hbar}{\|E - U_0\|} \left(n - \frac{1}{2} \right), \frac{\pi \hbar}{\|E - U_0\|} \left(n + \frac{1}{2} \right) \right]$$

Note that for $a = 1$ and $b = 0$, Eq. (21) reduces to

$$x(t) = ct + x_0 , \quad (22)$$

representing the relativistic equation of the photon's trajectories. As for the massive particle case we note that the RQTs of the photon pass through some points constituting nodes. All the trajectories cross the distance between two nodes at equal times. This establishment is also valid for the massive particle RQTs. However, the photon's RQTs in classically forbidden regions possess also nodes. This is not the case of the massive particle RQTs in these regions (see the precedent section).

To illustrate, we plotted in Fig. 3 many trajectories, of a free photon of energy 1.2 Mev (X ray), for different values of (a, b) . The nodes are clearly illustrated. The physical interpretation of these nodes will be exposed in Sec. 5.

4. The linear potential case

$x (\times 10^{-12} \text{ m})$

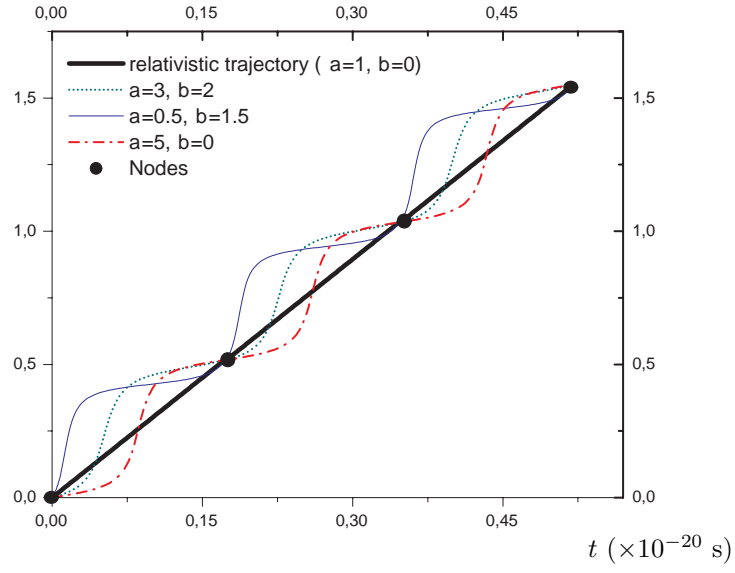


Fig. 3: Relativistic quantum trajectories for a free photon of energy $E = 1.2 \text{ MeV}$. We have chosen $x(t=0) = 0$.

Here, we investigate the motion of a massive particle (electron) under a potential of the form

$$V(x) = gx , \quad (23)$$

for which the Klein-Gordon equation takes the form

$$-c^2\hbar^2\frac{\partial^2\phi}{\partial x^2} + [m_0^2c^4 - (E - gx)^2]\phi(x) = 0 . \quad (24)$$

To establish the RQTs for the linear potential case, we can integrate the Eq. (8), where ϕ_1 and ϕ_2 are, now, two solutions of Eq. (24).

In this paper, we do not present the analytic solutions of Eq. (24), and in order to plot the RQTs, we approach the problem by numeric methods. We, first, integrate numerically Eq. (24) to obtain two independent solutions ϕ_1 and ϕ_2 , then, we plot the RQTs from Eq. (23). We opt in the two steps for the Euler integration method.

The RQTs for the linear potential are presented in Fig. 4. We choose $E = 2$ Mev, and $g = 1$ Kg.m.s. In Ref. [2], we have chosen $g = 10^{-9}$ Kg.m.s, which is very small compared with $g = 1$ Kg.m.s. We take this last value for relativistic problem to render the quantity gx , for quantum scales, of the same order as E , so that the Klein-Gordon equation do not reduces to the Schrödinger equation.

As we can notice from Fig. 4, the nodes are also present for the linear potential case. The distance between two adjacent nodes in Fig. 4 increase as the velocity decrease when it approaches the turning point (point where the velocity vanish). This note will be exposed in Sec. 5. Here we do not investigate the classically forbidden regions.

We would like to stress that as for the linear potential in quantum cases, we check that the positions of the nodes on x axis is related to the zeros of the solution of the Schrödinger equation ϕ_2 present in the denominator of the rapport in the expression (2) of the reduced action. This fact indicates that, the RQTs likes the QTs and it is obvious that the QT are a limit of the RQTs when $c \rightarrow \infty$.

5. De Broglie's wavelength

The most important idea that Secs. 2, 3 and 4 bring is the existence of node through which all RQTs pass, even the purely relativistic one. In this section, we link the distance between two adjacent nodes to the de Broglie's wavelength

$$\lambda = \frac{h}{p} \quad (25)$$

In Eq. (25), p is the relativistic momentum. For a particle moving under a constant potential

$$p = \frac{\sqrt{(E - U_0)^2 - m_0^2c^4}}{c} . \quad (26)$$

By replacing Eq. (26) in Eq. (25), we get

$$\lambda = \frac{hc}{\sqrt{(E - U_0)^2 - m_0^2c^4}} . \quad (27)$$

The distance between two in the case of a constant potential is

$$\Delta x_n = \frac{\pi\hbar c}{\sqrt{(E - U_0)^2 - m_0^2c^4}} . \quad (28)$$

$x (\times 10^{-12} \text{ m})$

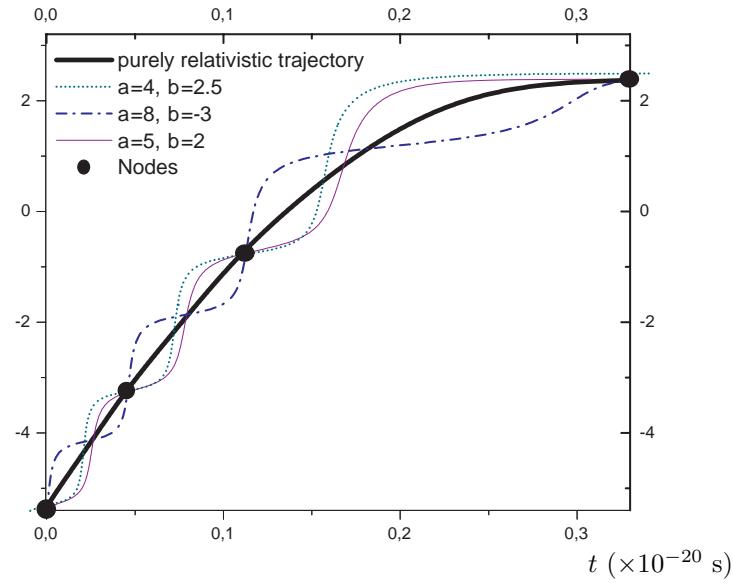


Fig. 4: Relativistic quantum trajectories for an electron of energy $E = 2 \text{ Mev}$ moving under a constant potential. We have chosen $x(t=0) = -5.4 \times 10^{-12} \text{ m}$.

From Eqs. (27) and (28) we get

$$\Delta x_n = \frac{\lambda}{2} . \quad (29)$$

Thus, the de Broglie's wavelength represent the double of the distance between two adjacent nodes. As we have presented in Ref. [2], we can generalize this definition for other potentials. Indeed, if we compute the mean value of $\partial S_0/\partial x$ between two adjacent nodes, and taking into account Eqs (11), (13) and (14) we find

$$\left\langle \frac{\partial S_0}{\partial x} \right\rangle = \frac{1}{\Delta x_n} \int_{x(t_n)}^{x(t_{n+1})} \frac{\partial S_0}{\partial x} dx = \frac{S_0(x(t_{n+1})) - S_0(x(t_n))}{\Delta x_n} = \frac{\sqrt{(E - U_0)^2 - m_0^2 c^4}}{c} , \quad (30)$$

which is equal to p (Eq. (26)). We propose to define a new wavelength after substituting p by

$$p = \left\langle \frac{\partial S_0}{\partial x} \right\rangle . \quad (31)$$

Then for any potential we can write, after using (2)

$$p = \frac{\pi \hbar}{\Delta x} , \quad (32)$$

with Δx is the distance between two adjacent nodes. If we substitute (32) in (25) we find

$$\Delta x = \frac{\lambda}{2} . \quad (33)$$

This relation links between the distance separating two adjacent nodes and the de Broglie's wavelength.

Conclusion

To conclude, we note that this article is a consequence of Ref. [3]. We have showed that for the RQTs the nodes still as an important result of our deterministic approach of quantum mechanics. These nodes are linked successfully to the de Broglie's wavelength. We also approach the photon's trajectories for a constant potential. For photons, other potentials will be studied in a next preparing paper.

By this results, we think that the generalization of our formalism [1, 2, 3] into relativistic systems is successful. Nevertheless, a generalization into more than one dimension of our new approach of quantum mechanics is an important step that we must investigate. In Ref. [9], we have started such as generalization by studying the QSHJE in three dimensions for symmetrical potentials.

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